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Modularity of special cycles on unitary Shimura varieties over CM-fields

by

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1. Introduction. Hirzebruch–Zagier [4] observed that the intersection number of special divisors on Hilbert modular surfaces generates a certain weight 2 elliptic modular form. Kudla–Millson generalized this work in [8], and they proved that special cycles on orthogonal (resp. unitary) Shimura varieties generate Siegel (resp. Hermitian) modular forms with coefficients in the cohomology group. Yuan–Zhang–Zhang [12] and Zhang [13] treated this problem in the Chow group in the case of orthogonal Shimura varieties and proved the modularity under a convergence assumption. Bruinier–Raum [3] showed the convergence. Kudla [7] and the author [10] generalized this problem to a certain orthogonal Shimura variety under the Beilinson–Bloch conjecture.

In this paper, we shall deal with the unitary case in the Chow group. Our problem is Conjecture 1.4. We give two solutions to this problem (Corollary 1.6 and Theorem 1.7). First, we prove Conjecture 1.4 for $e = 1$ unconditionally by using Bruinier's result [2]. On the other hand, for $e = 1$, Liu [9] solved Conjecture 1.4, i.e., proved the modularity of special cycles on unitary Shimura varieties in the Chow group, assuming the absolute convergence of the generating series. Recently, Xia [11] showed the modularity and absolute convergence of the generating series for $e = 1$. Our result in this paper gives another proof of Liu's result [9, Theorem 3.5]. For $e = 1$ and $r = 1$, the modularity of special divisors is proved in Theorem 1.5. To treat higher-codimensional cycles, we adopt the induction method [12]. Second, for $e > 1$, we show Conjecture 1.4 under the Beilinson–Bloch conjecture for

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orthogonal Shimura varieties. We reduce the problem to the orthogonal case ([7] and [10]), so we also need the Beilinson–Bloch conjecture for orthogonal Shimura varieties. We remark that we do not prove the absolute convergence of the generating series in this paper.

Before giving the statement of our results, we shall define some Shimura varieties.

1.1. Unitary Shimura varieties. Let d, e, and n be positive integers such that $e < d$. Let F be a totally real field of degree d with real embeddings $\sigma_1, \dots, \sigma_d$ and E be a CM extension of F. We write ∂_F for the different ideal of F. Let $(V_E, \langle\ ,\ \rangle)$ be a non-degenerate Hermitian space of dimension $n+1$ over E whose signature is $(n, 1)$ at $\sigma_1, \dots, \sigma_e$ and $(n+1, 0)$ at $\sigma_{e+1}, \dots, \sigma_d$.

For $i = 1, \dots, e$, let $V_{E,\sigma_i,\mathbb{C}} := V_E \otimes_{F,\sigma_i} \mathbb{C}$ and $D_i^E \subset \mathbb{P}(V_{E,\sigma_i,\mathbb{C}})$ be the Hermitian symmetric domain defined as

$$D_i^E := \{v \in V_{E,\sigma_i,\mathbb{C}} \backslash \{0\} \mid \langle v, v\rangle > 0\}/\mathbb{C}^\times.$$

We use

$$D_E := D_1^E \times \cdots \times D_e^E.$$

Let $\mathrm{U}(V_E)$ be the unitary group of V_E over F, which is a reductive group over F. We put $G := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V_E)$ and consider the Shimura varieties associated with the Shimura datum (G, D_E). Then, for any open compact subgroup $K_f^G \subset G(\mathbb{A}_f)$, the Shimura datum (G, D_E) gives a Shimura variety $M_{K_f^G}$ over $\mathbb{C}$, whose $\mathbb{C}$-valued points are given by

$$M_{K_f^G}(\mathbb{C}) = G(\mathbb{Q}) \backslash (D_E \times G(\mathbb{A}_f))/K_f^G.$$

Here, $\mathbb{A}_f$ is the ring of finite adèles of $\mathbb{Q}$. We remark that $M_{K_f^G}$ has a canonical model over a number field called the reflex field. Hence $M_{K_f^G}$ is canonically defined over $\overline{\mathbb{Q}}$, an algebraic closure of $\mathbb{Q}$ embedded in $\mathbb{C}$. By abuse of notation, in this paper, the canonical model of $M_{K_f^G}$ over $\overline{\mathbb{Q}}$ is also denoted by $M_{K_f^G}$. Then the Shimura variety $M_{K_f^G}$ is a projective variety over $\overline{\mathbb{Q}}$ since $0 < d - e$. It is a smooth variety over $\overline{\mathbb{Q}}$ if K_f^G is sufficiently small. In this paper, we assume that K_f^G is sufficiently small.

1.2. Orthogonal Shimura varieties. We define $V_F := V_E$, considered as an F-vector space, and $(\ ,\ ) := \mathrm{Tr}_{E/F}\langle\ ,\ \rangle$. Then $(V_F, (\ ,\ ))$ is a quadratic space of dimension $2n + 2$ over F whose signature is $(2n, 2)$ at $\sigma_1, \dots, \sigma_e$ and $(2n + 2, 0)$ at $\sigma_{e+1}, \dots, \sigma_d$. We define D_F similarly. We put $H := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(V_F)$ and define $N_{K_f^H}$ similarly for an open compact subgroup $K_f^H \subset H(\mathbb{A}_f)$. Let $L \subset V_F$ be a lattice, and L' the dual lattice. Now, we have a group embedding, $G \hookrightarrow H$. From here on, we assume that

$K_f^G = H(\mathbb{A}_f) \cap K_f^H$ so that

$$\iota\colon M_{K_f^G} \hookrightarrow N_{K_f^H}. \tag{1.1}$$

In this paper, we also assume that K_f^H is sufficiently small.

1.3. Special cycles on Shimura varieties. We shall define special cycles on unitary Shimura varieties. For $i = 1, \ldots, e$, let $\mathscr{L}_i \in \operatorname{Pic}(D_i^E)$ be the line bundle which is the restriction of $\mathscr{O}_{\mathbb{P}(V_{E_i,\sigma_i,\mathbb{C}})}(-1)$ to D_i^E. By pulling back to D_E, we get $p_i^*\mathscr{L}_i \in \operatorname{Pic}(D_E)$, where $p_i\colon D_E \to D_i^E$ are the projection maps. These line bundles descend to $\mathscr{L}_{K_f,i} \in \operatorname{Pic}(M_{K_f^G}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and thus we obtain $\mathscr{L} := \mathscr{L}_{K_f^G,1} \otimes \cdots \otimes \mathscr{L}_{K_f^G,e}$ on $M_{K_f^G}$.

We shall define special cycles following Kudla [6], [7]. Let $W \subset V_E$ be a totally positive subspace over E. We denote $G_W := \operatorname{Res}_{F/\mathbb{Q}} \mathrm{U}(W^\perp)$. Let $D_{W,E} := D_{W,1}^E \times \cdots \times D_{W,e}^E$ be the Hermitian symmetric domain associated with G_W, where

$$D_{W,i}^E := \{w \in D_i^E \mid \forall v \in W_{\sigma_i},\ \langle v, w\rangle = 0\} \quad (1 \le i \le e).$$

Then we have an embedding of Shimura data $(G_W, D_{W,E}) \hookrightarrow (G, D_E)$. For any open compact subgroup $K_f^G \subset G(\mathbb{A}_f)$ and $g \in G(\mathbb{A}_f)$, we have an associated Shimura variety $M_{gK_f^G g^{-1},W}$ over $\mathbb{C}$:

$$M_{gK_f^G g^{-1},W}(\mathbb{C}) = G_W(\mathbb{Q})\backslash(D_{W,E} \times G_W(\mathbb{A}_f))/(gK_f^G g^{-1} \cap G_W(\mathbb{A}_f)).$$

Assume that K_f^G is neat so that the morphism

$$M_{gK_f^G g^{-1},W}(\mathbb{C}) \to M_{K_f^G}(\mathbb{C}), \quad [\tau, h] \mapsto [\tau, hg],$$

is a closed embedding [7, Lemma 4.3]. Let $Z^G(W, g)_{K_f^G}$ be the image of this morphism. We consider $Z^G(W, g)_{K_f^G}$ to be an algebraic cycle of codimension $e \dim_F W$ on $M_{K_f^G}$ defined over $\overline{\mathbb{Q}}$.

For any positive integer r and $x = (x_1, \ldots, x_r) \in V_E^r$, let $U(x)$ be the E-subspace of V_E spanned by $x_1, \ldots, x_r$. We define the *special cycle* in the Chow group

$$Z^G(x, g)_{K_f^G} \in \mathrm{CH}^{er}(M_{K_f^G})_{\mathbb{C}} := \mathrm{CH}^{er}(M_{K_f^G}) \otimes_{\mathbb{Z}} \mathbb{C}$$

by

$$Z^G(x, g)_{K_f^G} := Z^G(U(x), g)_{K_f^G}(\mathrm{c}_1(\mathscr{L}_{K_f^G,1}^\vee) \cdots \mathrm{c}_1(\mathscr{L}_{K_f^G,e}^\vee))^{r-\dim U(x)}$$

if $U(x)$ is totally positive. Otherwise, we put $Z^G(x, g)_{K_f^G} := 0$.

For a Bruhat–Schwartz function $\phi_f \in \boldsymbol{S}(V_E(\mathbb{A}_f)^r)^{K_f^G}$ that is K_f^G-invariant, *Kudla's generating function* is defined to be the following formal power series with coefficients in $\mathrm{CH}^{er}(M_{K_f^G})_{\mathbb{C}}$ in the variable $\tau = (\tau_1, \ldots, \tau_d) \in (\mathscr{H}_r)^d$:

$$Z^G_{\phi_f}(\tau) := \sum_{x \in G(\mathbb{Q})\backslash V_E^r} \sum_{g \in G_x(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f^G} \phi_f(g^{-1}x) Z^G(x,g)_{K_f^G} q^{T(x)}.$$

Here, $G_x \subset G$ is the stabilizer of x, $\mathscr{H}_r$ is the Siegel upper half-plane of genus r, $T(x)$ is the moment matrix $\frac{1}{2}((x_i, x_j))_{i,j}$, and

$$q^{T(x)} := \exp\Big(2\pi\sqrt{-1}\sum_{i=1}^{d} \mathrm{Tr}\, \tau_i \sigma_i T(x)\Big).$$

For a $\mathbb{C}$-linear map $\ell\colon\ \mathrm{CH}^{er}(M_{K_f^G})_{\mathbb{C}} \to \mathbb{C}$, we put

$$\ell(Z^G_{\phi_f})(\tau) := \sum_{x \in G(\mathbb{Q})\backslash V_E^r} \sum_{g \in G_x(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f^G} \phi_f(g^{-1}x) \ell(Z^G(x,g)_{K_f^G}) q^{T(x)},$$

which is a formal power series with complex coefficients in the variable $\tau \in (\mathscr{H}_r)^d$. We define $Z^H_{\phi_f}(\tau)$ similarly.

Remark 1.1. We explain that $Z^H_{\phi_f}(\tau)$ is an analogue of a theta function. For a totally real definite matrix $\beta \in M_r(F)$, let $\Omega_\beta := \{x \in V_F^r \mid T(x) = \beta\}$, and we consider the Fourier expansion with respect to β. Now we choose β such that $\Omega_\beta \neq \emptyset$ and fix $x_0 \in \Omega_\beta(F)$. For $\xi_j \in H(\mathbb{A}_f)$, we have

$$\mathrm{Supp}(\phi_f) \cap \Omega_\beta(\mathbb{A}_f) = \coprod_{j=1}^{\ell} K_f^H \cdot \xi_j \cdot x_0,$$

and we put

$$Z^H(\beta, \phi_f)_{K_f^H} := \sum_{j=1}^{\ell} \phi_f(\xi_j^{-1} \cdot x_0) Z^H(x_0, \xi_j)_{K_f^H}.$$

Then $Z^H_{\phi_f}(\tau)$ becomes

$$Z^H_{\phi_f}(\tau) = \sum_{\beta \geq 0} Z^H(\beta, \phi_f)_{K_f^H} q^\beta$$

and by adding Kudla–Millson forms and Gaussian functions, this is exactly a theta function in the cohomology group. For details, see [6].

Before stating our goal, we have to clarify the notion of "modular".

Definition 1.2. Let V be a vector space over $\mathbb{C}$ and f be a formal power series with coefficients in V. We say f is *modular* if for any $\mathbb{C}$-linear map $\ell\colon V \to \mathbb{C}$ such that $\ell(f)$ is absolutely convergent, $\ell(f)$ is modular.

1.4. The Beilinson–Bloch conjecture. To state the main theorem for $e > 1$, we have to introduce the Beilinson–Bloch conjecture because we need modularity for the orthogonal case, and this was proved under the Beilinson–Bloch conjecture in [7] and [10].

Let X be a smooth variety over $\overline{\mathbb{Q}}$ and

$$\mathrm{cl}^m : \mathrm{CH}^m(X) \to H^{2m}(X, \mathbb{Q}) := H^{2m}(X(\mathbb{C}), \mathbb{Q})$$

be the mth cycle map. On the other hand, we have the mth intermediate Jacobian $J^{2m-1}(X)$ of X defined by the Hodge structure of X (see before [10, Conjecture 1.2]). Then there exists the mth higher Abel–Jacobi map

$$\mathrm{AJ}^m : \mathrm{Ker}(\mathrm{cl}^m) \otimes \mathbb{Q} \to J^{2m-1}(X) \otimes \mathbb{Q}.$$

The Beilinson–Bloch conjecture claims that AJ^m is injective. Consequently, if $H^{2m-1}(X, \mathbb{Q}) = 0$, then under the Beilinson–Bloch conjecture for m, the map

$$\mathrm{cl}^m_{\mathbb{Q}} : \mathrm{CH}^m(X) \otimes \mathbb{Q} \to H^{2m}(X, \mathbb{Q}) := H^{2m}(X(\mathbb{C}), \mathbb{Q})$$

is injective. See [10] for the detailed claim of the Beilinson–Bloch conjecture.

Remark 1.3. For the rest of this paper, "the Beilinson–Bloch conjecture for m" means that $\mathrm{cl}^m_{\mathbb{Q}}$ is injective if $H^{2m-1}(X, \mathbb{Q}) = 0$. We will assume this conjecture for orthogonal Shimura varieties because the modularity of the generating series for them (see [7] or [10]), in the Chow groups used the result by Kudla–Milson [8], which asserts the modularity of the generating series in the cohomology group. Namely, Kudla and the author obtained the modularity with the Chow group coefficients under the Beilinson–Bloch conjecture. We will deduce the modularity for unitary Shimura varieties from their results.

1.5. Main results. For notations, see Subsections 1.2 and 1.3. In the context of *Kudla's modularity conjecture*, our problem is as follows.

Conjecture 1.4. *The generating series* $Z^G_{\phi_f}(\tau)$ *is a Hilbert-Hermitian modular form of weight* $n+1$ *and genus* r*.*

We give two partial solutions to this problem: Corollary 1.6 and Theorem 1.7.

First, we can prove the modularity of the generating series of special divisors by using the regularized theta lift on orthogonal groups.

Theorem 1.5 (Theorem 3.1). *Assume that* $e = 1$ *and* $r = 1$*. Then* $Z^G_{\phi_f}(\tau)$ *is a Hilbert-Hermitian modular form for* $\mathrm{SU}(1,1)$ *of weight* $n+1$ *under the assumption that the series converges absolutely.*

Theorem 1.5 generalizes [5, Theorem 10.1]. We can prove a stronger result by induction on r [12]; see Corollary 1.6. It does not follow immediately from [5] or Theorem 1.5 that $Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(1,1)$, i.e., Theorem 1.5 shows only the $\mathrm{SU}(1,1)$-modularity of $Z^G_{\phi_f}(\tau)$. However, we can show the $\mathrm{U}(1,1)$-modularity of $Z^G_{\phi_f}(\tau)$ by proving termwise modularity. This means that we can show the modularity of $Z^G_{\phi_f}(\tau)$ for the parabolic subgroup P_1 and a specific element w_1 defined in Section 3. On the other hand, P_1 and w_1 generate $\mathrm{U}(1,1)$, and we already know the modularity for $w_1 \in \mathrm{SU}(1,1)$ from Theorem 1.5, so the problem reduces to proving the modularity for P_1. For the proof of modularity for P_1, see [9], [10], and [12]. By combining the above modularity and induction on r, we can prove the modularity of special cycles of a higher codimension.

Corollary 1.6 (Corollary 3.2). *Assume that $e = 1$. Then $Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ under the assumption that the series converges absolutely.*

This gives another proof of Theorem 1.7 for $e = 1$ and of [9, Theorem 3.5]. This is shown unconditionally, unlike Theorem 1.7.

Now, we state the theorem for $e > 1$. Recall that $G := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V_E)$ is the unitary group associated with a Hermitian space V_E over a CM field E, and for a Bruhat–Schwartz function $\phi_f \in \boldsymbol{S}(V_E(\mathbb{A}_f)^r)^{K^G_f}$, our generating series $Z^G_{\phi_f}(\tau)$ is defined as follows with coefficients in $\mathrm{CH}^{er}(M_{K^G_f})_{\mathbb{C}}$ in the variable $\tau = (\tau_1, \dots, \tau_d) \in (\mathscr{H}_r)^d$:

$$Z^G_{\phi_f}(\tau) := \sum_{x \in G(\mathbb{Q}) \backslash V_E^r} \; \sum_{g \in G_x(\mathbb{A}_f) \backslash G(\mathbb{A}_f)/K^G_f} \phi_f(g^{-1}x) Z^G(x,g)_{K^G_f} q^{T(x)}.$$

Our main result in this paper is as follows.

Theorem 1.7 (Theorem 4.1). *$Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ under the Beilinson–Bloch conjecture for $m = e$ with respect to orthogonal Shimura varieties and under the assumption that the series converges absolutely for $e > 1$.*

Remark 1.8. We assume the Beilinson–Bloch conjecture for $m = e$ for $N_{K^H_f}$ when $2n \geq 3$, i.e., $n > 1$. When $n = 1$, we need to assume the Beilinson–Bloch conjecture for $m = e$ for a larger orthogonal Shimura variety $N'_{K^H_f}$ including $N_{K^H_f}$; see [10, Theorem 1.6]. For the precise statement of the Beilinson–Bloch conjecture, see [10, Section 1.2].

Remark 1.9. Kudla [7] and the author [10] proved the modularity of the generating series associated with orthogonal Shimura varieties for $e > 1$. These results are shown by using Kudla–Millson's cohomological coefficients result [8] and reducing the problem to this cohomological case under the Beilinson–Bloch conjecture for orthogonal Shimura varieties. Therefore one might think that the modularity of the generating series associated with unitary Shimura varieties could also be proved in the same way, but the Hodge numbers appearing in the cohomology of unitary Shimura varieties do not seem to vanish [7, Remark 1.2].

Historically, for unitary Shimura varieties, Kudla–Millson [8] studied the cohomological coefficients case. In the Chow group, Hofmann [5] showed the $\mathrm{SL}_2(\cong \mathrm{SU}(1,1))$-modularity of the generating series over imaginary quadratic fields for $r = 1$, $e = 1$, and Liu [9] showed Hermitian modularity for $e = 1$, assuming the absolute convergence of the generating series. We generalize their work. On the other hand, Xia [11] showed Liu's result without assuming the absolute convergence of the generating series. He uses the formal Fourier–Jacobi series method similar to the work over $\mathbb{Q}$ of Bruinier–Westerholt–Raum [3].

Theorem 1.5 and Corollary 1.6 are included in Theorem 1.7 under the Beilinson–Bloch conjecture, but we give another proof working only for $r = 1$, using regularized theta lifts.

We can also restate the result using Kudla's modularity conjecture for orthogonal Shimura varieties as follows.

Corollary 1.10. *$Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r, r)$ of weight $n + 1$, assuming the modularity of the generating series of special cycles on orthogonal Shimura varieties for $r = 1$ and absolute convergence of the series $Z^G_{\phi_f}(\tau)$ for $e > 1$.*

We explain in Section 4.4 why we only assume the modularity for $r = 1$ on orthogonal Shimura varieties.

1.6. Outline of the proof of Theorems 1.5 and 1.7. As an application of the modularity of special cycles on orthogonal Shimura varieties proved by using regularized theta lifts, we can prove Theorem 1.5 and Corollary 1.6. This is another proof of [9, Theorem 3.5] for the special divisors case. Theorem 1.7 can be reduced to the orthogonal case (see [7] and [10]), so we have to assume the Beilinson–Bloch conjecture for orthogonal Shimura varieties, and this is our solution to Conjecture 1.4.

1.7. Outline of this paper. In Section 2, we review the modularity of the generating series of special cycles on orthogonal Shimura varieties. In Section 3, we prove modularity for $e = 1$. In Section 4, we establish the Hermitian modularity of special cycles for $e > 1$ under the Beilinson–Bloch conjecture for orthogonal Shimura varieties.

2. Modularity on orthogonal groups. In this section, we shall recall Bruinier's work [2]. He constructed regularized theta lifts on orthogonal groups and showed the modularity of special cycles on orthogonal Shimura varieties.

Throughout this section, let $L \subset V_F$ be an even $\mathscr{O}_F$-lattice and L' be the $\mathbb{Z}$-dual lattice of L with respect to $\mathrm{Tr}_{F/\mathbb{Q}}(\ ,\ )$. Let $\hat{\mathbb{Z}} := \prod_{p<\infty} \mathbb{Z}_p$, and we define $\hat{L} := L \otimes \hat{\mathbb{Z}}$. We have $L'/L \cong \hat{L}'/\hat{L}$, so for $\mu \in L'/L$, let $1_\mu \in \boldsymbol{S}(V_F(\mathbb{A}_{F,f}))$ be the characteristic function associated with $\mu + \hat{L}$. In the current section, we assume that $r = 1$ and $n > 2$.

2.1. Regularized theta lifts on orthogonal groups. We review the results of [2]. Let

$$k := (k_1, k_2, \dots, k_d) = (1-n, 1+n, \dots, 1+n) \in \mathbb{Z}^d$$

and $s_0 := 1 - k_1 = n$. We call k a *weight* and define the *dual weight* κ to be

$$\kappa := (2-k_1, k_2 \dots, k_d) = (1+n, 1+n, \dots, 1+n) \in \mathbb{Z}^d.$$

We use Kummer's confluent hypergeometric function

$$M(a,b,z) := \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}, \qquad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)},$$

for $a, b, z \in \mathbb{C}$, and Whittaker functions

$$M_{\nu,t}(z) := e^{-z/2} z^{1/2+t} M(1/2+t-\nu, 1+2t, z) \qquad (t, \nu \in \mathbb{C}),$$
$$\mathscr{M}_s(v_1) := |v_1|^{-k_1/2} M_{\mathrm{sgn}(v_1)k_1/2, s/2}(|v_1|) e^{-v_1/2} \qquad (s \in \mathbb{C},\ v_1 \in \mathbb{R}).$$

Now, we define the Whittaker forms

$$f_{m,\mu}(\tau, s) := C(m,k,s)\mathscr{M}_s(-4\pi m_1 v_1) \exp(-2\pi\sqrt{-1}\,\mathrm{Tr}(m\overline{\tau}))1_\mu \qquad (m_i := \sigma_i(m)),$$

where $\mu \in L'/L \cong \hat{L}'/\hat{L}$ and 1_μ is the characteristic function associated with $\mu + \hat{L}$. Here, $C(m,k,s)$ is a normalizing factor,

$$C(m,k,s) := \frac{(4\pi m_2)^{k_2-1} \cdots (4\pi m_d)^{k_d-1}}{\Gamma(s+1)\Gamma(k_2-1)\cdots\Gamma(k_d-1)}.$$

We define, for $\tau \in (\mathscr{H}_1)^d$, the function

$$\begin{aligned} f_{m,\mu}(\tau) &:= f_{m,\mu}(\tau, s_0) \\ &= C(m,k,s_0)\Gamma(2-k_1)\left(1 - \frac{\Gamma(1-k_1, 4\pi m_1 v_1)}{\Gamma(1-k_1)}\right) \\ &\quad \cdot e^{4\pi m_1 v_1} \exp(-2\pi\sqrt{-1}\,\mathrm{Tr}(m\overline{\tau}))1_\mu. \end{aligned}$$

For $m \in F$, $m \gg 0$ means $m_i := \sigma_i(m) > 0$ for all i, and ∂_F denotes the different ideal of a totally real field F. Note that we consider a finite $\mathscr{O}_F$-module L'/L equipped with a quadratic form $(\ ,\ )/2$ which takes values in $F/\partial^{-1}\mathscr{O}_F$ since we assume that L is even.

Definition 2.1. A *Whittaker form of weight* k is a finite linear combination of the functions $f_{m,\mu}(\tau, s)$ for $\mu \in L'/L, m \in (\mu,\mu)/2 + \partial^{-1}\mathscr{O}_F$ and $m \gg 0$. A *harmonic Whittaker form of weight* k is a Whittaker form with $s = s_0$, i.e., a function of the form

$$\sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu) f_{m,\mu}(\tau)$$

for $c(m,\mu) \in \mathbb{C}$. Here, the second sum runs $m \in (\mu,\mu)/2 + \partial^{-1}\mathscr{O}_F$. Let $H_{k,\overline{\rho_L}}$ be the $\mathbb{C}$-vector space consisting of harmonic Whittaker forms of weight k.

Note that in the above definition, the weight k is used in the definition of the normalizing factor $C(m,k,s)$ and $s_0 := 1 - k_1$.

Remark 2.2. Here, ρ_L is a lattice model of the Weil representation of the metaplectic group $\mathrm{Mp}_2(\hat{\mathscr{O}}_F)$, and $f_{m,\mu}$ satisfies a certain modularity condition on ρ_L and a certain differential equation. For details, see [2, Chapter 4].

Under our assumption on $n > 2$ and $\kappa_j \geq 2$ for all j, there is a surjective map $\xi_k \colon H_{k,\overline{\rho_L}} \to S_{\kappa,\rho_L}$ [2, Proposition 4.3]. Here, S_{κ,ρ_L} is the space of Hilbert modular forms of weight κ and type ρ_L. Let $M^!_{k,\overline{\rho_L}}$ be the kernel of this map, and we call elements of this space *weakly holomorphic Whittaker forms of weight* k. Hence, there is an exact sequence

$$0 \to M^!_{k,\overline{\rho_L}} \to H_{k,\overline{\rho_L}} \xrightarrow{\xi_k} S_{\kappa,\rho_L} \to 0.$$

This exact sequence and the following are analogues of classical ones. See Borcherds [1]. This pairing is non-degenerate, so a non-degenerate pairing is induced between $H_{k,\overline{\rho_L}}/M^!_{k,\overline{\rho_L}}$ and S_{k,ρ_L}, defined by

$$\{g, f\} := (g, \xi_k(f))_{\mathrm{Pet}}$$

for the Petersson inner product on S_{k,ρ_L}. We recall an explicit formula for the pairing $\{\ ,\ \}$.

Proposition 2.3 ([2, Proposition 4.5]). *For* $g \in S_{\kappa,\rho_L}$ *and* $f \in H_{k,\overline{\rho_L}}$ *with Fourier expansions*

$$g = \sum_{\nu \in L'/L} \sum_{n >> 0} b(n,\nu) \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(n\tau)) 1_\nu,$$

$$f = \sum_{\mu \in L'/L} \sum_{m >> 0} c(m,\mu) f_{m,\mu}(\tau),$$

we have

$$\{g, f\} = \sum_{\mu \in L'/L} \sum_{m >> 0} c(m,\mu) b(m,\mu).$$

We remark that Whittaker forms are analogues of Maass forms; see [2, Section 4.1].

For $f = \sum_\mu \sum_m c(m,\mu) f_{m,\mu}(\tau) \in H_{k,\overline{\rho_L}}$, we define

$$Z(f) := \sum_\mu \sum_m c(m,\mu) Z^H(m,\mu)_{K_f^H}.$$

Let $I := \operatorname{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$ and χ_V be a quadratic character of $\mathbb{A}_F^\times / F^\times$ associated with V, given by

$$\chi_V(x) := (x, (-1)^{\ell(\ell-1)/2} \det(V))_F \quad (\ell := 2n+2).$$

We review the definition of Eisenstein series [2, Section 6.2]. Let $Q \subset H$ be the parabolic subgroup consisting of upper triangular matrices, and let $s \in \mathbb{C}$. We take a standard section $\varPhi \in I(s,\chi) := \operatorname{Ind}_Q^H \chi_V |\cdot|^s$. Then we have the Eisenstein series

$$E(g,s,\varPhi) := \sum_{\gamma \in I(F) \backslash H(F)} \varPhi(\gamma g),$$
$$E(\tau,s,\ell;\varPhi_f) := v^{-\ell/2} E(g_\tau, s, \varPhi_f \otimes \varPhi_\infty^\ell),$$

where $g_\tau \in \mathrm{Mp}_2(\mathbb{R})^d$ satisfies $g_\tau(\sqrt{-1},\ldots,\sqrt{-1}) = \tau \in \mathscr{H}^d$, and $\varPhi_\infty^\ell$ is defined in [2, Chapter 6]. Let $1_\mu \in \boldsymbol{S}(V_F(\mathbb{A}_{F,f}))$ be the characteristic function associated with $\mu + \hat{L}$ for $\mu \in L'/L \cong \hat{L}'/\hat{L}$. Here, the Weil representation gives an intertwining operator between the space of Bruhat–Schwartz functions and the space of standard sections at $s = s_0$:

$$\lambda = \lambda \otimes \lambda_f \colon \boldsymbol{S}(V(\mathbb{A}_F)) \to I(s_0, \chi_V).$$

We obtain a vector valued Eisenstein series of weight ℓ with respect to ρ_L by taking

$$E_L(\tau,s,\ell) := \sum_{\mu \in L'/L} E(\tau,s,\ell;\lambda_f(1_\mu)) 1_\mu.$$

We get the Fourier expansion of the Eisenstein series at ∞:

$$E_L(\tau,\kappa) := E_L(\tau,s_0,\kappa) = 1_0 + \sum_{\mu \in L'/L} \sum_{m>>0} B(m,\mu) \exp(2\pi\sqrt{-1}\operatorname{Tr}(m\tau)) 1_\mu.$$

We define

$$B(f) := \sum_{\mu \in L'/L} \sum_{m>>0} c(m,\mu) B(m,\mu)$$

for a harmonic Whittaker form $f = \sum_\mu \sum_m c(m,\mu) f_{m,\mu}$. Note that

$$B(f) = \{E_L(\tau,\kappa), f\}.$$

The following theorem is the regularized theta lift over totally real fields, proved by Bruinier [2, Theorem 1.3].

Theorem 2.4 ([2, Theorem 6.8]). *Let* $f \in M^!_{k,\overline{\rho_L}}$ *be a weakly holomorphic Whittaker form of weight* k *for* $\Gamma = \mathrm{SL}_2(\mathscr{O}_F) \subset I(\mathbb{R}) = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2(\mathbb{R})$ *whose coefficients* $c(m,\mu)$ *are integral. Then there exists a meromorphic modular form* $\Psi_f(\tau, g)$ *for* $H(\mathbb{Q})$ *of level* K_f^H *such that*

(1) *the weight of* Ψ *is* $-B(f)$,
(2) $\operatorname{div}\Psi = Z(f)$.

2.2. Modularity of special divisors on orthogonal groups. Now, we review the modularity of special divisors on orthogonal Shimura varieties. To state the theorem, we need to prepare the generating series for orthogonal Shimura varieties. From [2], recall that for x_0 taken in Remark 1.1 and for a totally real element $m = \langle x_0, x_0\rangle/2 \gg 0$ in F, we define

$$Z^H(m,\phi_f)_{K_f^H} := \sum_{h\in H_{x_0}\backslash H(\mathbb{A}_{F,f})/K_f^H} \phi_f(h^{-1}x_0) Z^H(x_0,h),$$

and

$$\begin{aligned} A^0(\tau) :=& \sum_{\mu\in L'/L} -c_1(\mathscr{L})1_\mu \\ &+ \sum_{\mu\in L'/L}\sum_{m\gg 0}\big(Z^H(m,1_\mu)_{K_f^H} + B(m,\mu)c_1(\mathscr{L})\big)q^m 1_\mu, \\ A(\tau,\phi_f) :=& -c_1(\mathscr{L}) + \sum_{m>>0} Z^H(m,\phi_f)_{K_f^H} q^m, \\ A(\tau) :=& \sum_{\mu} A(\tau,1_\mu)1_\mu \\ =& \sum_{\mu\in L'/L} -c_1(\mathscr{L})1_\mu + \sum_{\mu\in L'/L}\sum_{m>>0} Z^H(m,1_\mu)_{K_f^H} q^m 1_\mu. \end{aligned}$$

Note that since $r=1$, x_0 is an element of V_F, so that the phrase "m is totally real" makes sense and corresponds to $\beta \geq 0$ in Remark 1.1.

We want to show the modularity of $Z_{\phi_f}(\tau)$, but first we will prove the modularity of $A^0(\tau)$ (see Remark 2.6). We remark that $A(\tau,\phi_f) = Z^H_{\phi_f}(\tau)$. The following theorem was proved by Bruinier [2].

Theorem 2.5 ([2, Theorem 7.1, Proposition 7.3]). *For any* $n > 0$,

$$A^0(\tau) \in S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(N_{K_f^H}).$$

Remark 2.6. We know

$$A^0(\tau) = A(\tau) + c_1(\mathscr{L})E_L(\tau,\kappa)$$

by [2, Remark 6.5]. Combining this with Theorem 2.5, we also get

$$A(\tau) \in S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(N_{K_f^H}).$$

3. Modularity of special cycles on unitary groups for $e = 1$.

3.1. Divisors case

THEOREM 3.1. *Assume that $e = 1$ and $r = 1$. Then $Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{SU}(1,1)$ of weight $n+1$ under the assumption that the series converges absolutely.*

Proof. First, we show the modularity of $Z^H_{\phi_f}(\tau)$. Since, ϕ_f is a locally constant, compactly supported function, we can write it as $\phi_f = \sum_{\mu \in L'/L} e_\mu 1_\mu$ for some $e_\mu \in \mathbb{C}$ and $\mu \in L'/L$. Recall that

$$S_L := \bigoplus_{\mu \in L'/L} \mathbb{C}1_\mu \subset \boldsymbol{S}(V(\mathbb{A}_{F,f})),$$

so that we define

$$\delta : S_L \to \mathbb{C}, \qquad \sum_{\mu \in L'/L} c_\mu 1_\mu \mapsto \sum_{\mu \in L'/L} c_\mu e_\mu.$$

Then we have

$$\delta \colon S_L \otimes \mathrm{CH}^1(N_{K^H_f})_{\mathbb{C}}[[q]] \supset S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(N_{K^H_f})_{\mathbb{C}}[[q]] \to \mathrm{CH}^1(N_{K^H_f})_{\mathbb{C}}[[q]],$$
$$\sum_{\mu \in L'/L} \sum_m b(m,\mu) 1_\mu \otimes Z_{m,\mu} q^m \mapsto \sum_{\mu \in L'/L} \sum_m b(m,\mu) e_\mu Z_{m,\mu} q^m,$$

where $\sum_{\mu \in L'/L} \sum_m b(m,\mu) 1_\mu q^m \in S_{\kappa,\rho_L}$ and $Z_{m,\mu} \in \mathrm{CH}^1(N_{K^H_f})_{\mathbb{C}}$. Note that we consider these two spaces as formally defined, without assuming absolute convergence. Then $\delta(A(\tau)) = Z^H_{\phi_f}(\tau)$, because from the definition of the generating series and Remark 1.1,

$$Z^H_{\phi_f}(\tau) = \sum_{m >> 0} Z^H(m, \phi_f)_{K^H_f} q^m,$$

where $q^m := \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(m\tau))$ Hence, this is formally modular in the sense of Definition 1.2 in view of Theorem 2.5 and Remark 2.6. See also [2, Section 2.3].

On the other hand, by [9, Corollary 3.4], we have $\iota^\star Z^H_{\phi_f}(\tau) = Z^G_{\phi_f}(\tau)$. Therefore, by the modularity of $Z^H_{\phi_f}(\tau)$, the generating series $Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{SU}(1,1)$ under the assumption that the series converges absolutely. Since the weight of $Z^H_{\phi_f}(\tau)$ is $n+1$, this finishes the proof. ■

This gives a proof of Theorem 1.5. Note that to prove the modularity of $Z^G_{\phi_f}(\tau)$ for $n > 1$, we use the perfect pairing presented in Proposition 2.3. For $n = 1$, we use an embedding trick. For more details, see [2] or [10].

3.2. General r case. To show Hermitian modularity, we reduce the problem to the generators of the associated unitary group. Now, the indefinite unitary group $\mathrm{U}(r,r)$ is generated by the parabolic subgroup $P_r(F) = M_r(F)N_r(F)$ and $w_{r,r-1}$, where

$$M_r(F) := \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\overline{a^{-1}} \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_r(E) \right\},$$

$$N_r(F) := \left\{ n(u) = \begin{pmatrix} 1_r & u \\ 0 & 1_r \end{pmatrix} \;\middle|\; u \in \mathrm{Her}_r(E) \right\},$$

$$w_{r,r-1} := \begin{pmatrix} 1_{r-1} & 0 & 0_{r-1} & 0 \\ 0 & 0 & 0 & 1 \\ 0_{r-1} & 0 & 1_{r-1} & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

See [9, proof of Theorem 3.5]. We put $w_1 := w_{1,0}$. By induction on r, we get the following result.

Corollary 3.2. *Assume $e = 1$. Then $Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ provided that the series converges absolutely.*

Proof. To prove that $Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $r = 1$, we note that we already know the modularity for $\mathrm{SU}(1,1)$ from Theorem 3.1. Therefore, in particular, we know the modularity for $w_1 \in \mathrm{SU}(1,1)$. Hence, it suffices to prove the modularity for the parabolic subgroup $P_1 \subset \mathrm{U}(1,1)$ because $\mathrm{U}(1,1)$ is generated by P_1 and $w_1 = w_{1,0}$. We can prove the invariance under P_1 in the same way as in [9] or [10]. This finishes the proof of the corollary for $r = 1$. For $r > 1$, we use induction on r. More specifically, for any r, we can prove the modularity for P_r, i.e.,

$$\omega_f(n(u)_f g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^G(x)_{K^G_f} = \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^G(x)_{K^G_f},$$
$$\omega_f(m(a)_f g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^G(x)_{K^G_f} = \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^G(x)_{K^G_f},$$

for any $u \in \mathrm{Her}_r(F)$ and $a \in \mathrm{GL}_r(F)$. This will also be done in more detail in Section 4.2. By using the modularity for w_1 and $r = 1$, we can prove the modularity for $w_{r,r-1}$ when $r > 1$ in the same way as in Section 4.3, and we already know the w_1-modularity. We will show the induction step in Section 4.3. ∎

This shows the modularity of special cycles on a unitary Shimura variety for $e = 1$ (Theorem 1.6) and gives another proof of Liu's theorem [9, Theorem 3.5].

4. General e case

4.1. Weil representations. Let $\psi\colon E\backslash\mathbb{A}_E \to \mathbb{C}^\times$ be the composite of the trace map $E\backslash\mathbb{A}_E \to \mathbb{Q}\backslash\mathbb{A}$ and the usual additive character

$$\mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times, \qquad (x_v)_v \mapsto \exp\Big(2\pi\sqrt{-1}\Big(x_\infty - \sum_{v<\infty} \overline{x_v}\Big)\Big),$$

where $\overline{x_v}$ is the class of x_v in $\mathbb{Q}_p/\mathbb{Z}_p$.

Let $(W, (\ ,\ ))$ be a Hermitian space of dimension $2r$ over E whose signature is (r,r) so that $\mathrm{U}(W) = \mathrm{U}(r,r)$. Then we get a symplectic vector space $\mathscr{W} := \mathrm{Res}_{E/F}(V_E \otimes_E W)$ with the skew-symmetric form $\mathrm{Tr}_{E/F}(\langle\ ,\ \rangle \otimes (\ ,\ ))$. Let $\mathrm{Sp}(\mathscr{W})$ be the symplectic group and $\mathrm{Mp}(\mathscr{W})$ be its metaplectic $\mathbb{C}^\times$ covering group. Then we get the Weil representations ω_f and $\omega_{\mathbb{A}}$, the actions of $\mathrm{Mp}(\mathscr{W})(\mathbb{A}_f)$ on $\boldsymbol{S}(V(\mathbb{A}_{F,f})^r)$ and $\mathrm{Mp}(\mathscr{W})(\mathbb{A})$ on $\boldsymbol{S}(V(\mathbb{A}_F)^r)$.

Now, we state the second solution to Conjecture 1.4.

Theorem 4.1. *Assuming absolute convergence for $e > 1$, $Z^G_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ under the Beilinson–Bloch conjecture for orthogonal Shimura varieties for $m = e$ provided that the series in the orthogonal case converges absolutely.*

We reduce Theorem 4.1 to the orthogonal case, so we have to assume the Beilinson–Bloch conjecture for orthogonal Shimura varieties. The strategy is as follows. For general e, we can prove the modularity for P_r for any r by direct calculation. We can also show the modularity for $w_{r,r-1}$ when $r > 1$, assuming the modularity for $w_1 = w_{1,0}$ and $r = 1$. Hence, the problem is the modularity for w_1 for $r = 1$ and general e. We treat this problem by embedding unitary Shimura varieties into orthogonal varieties, studied in [5]. In the orthogonal cases, the modularity of the generating series is proved by [7] or [10] under the Beilinson–Bloch conjecture. We remark that when $e = 1$, the modularity for w_1 is solved by Corollary 3.2, followed by the modularity for $\mathrm{SU}(1,1)$ using the regularized theta lifts. For the precise statement of the Beilinson–Bloch conjecture, see [10, Section 1.2].

From [12], we get the following expression for the generating series for the unitary group G:

$$Z^G_{\phi_f}(\tau) = \sum_{\substack{x \in K^G_f \backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2 \in Ex} \sum_{\substack{y_1 \in K^G_{f,x} \backslash x^\perp \\ \text{admissible}}} \phi_f(x, y_i + y_2) Z^G(y_1)_{K_{f,x}} q^{T(x, y_1+y_2)},$$

where $K^G_{f,x}$ is the stabilizer of x and let $\widehat{V}_E := V_E \otimes \mathbb{A}_f$. Here, for the notion of "admissible" and the definition of the special cycles $Z^G(x)_{K_f}$, see [9, Lemma 3.1], [10, Lemma 2.1], or [12, Lemma 2.1]. Let $\varphi_+(x) = \exp(-\pi \,\mathrm{Tr}\, T(x))$ be the Gaussian. We extend the definition of $Z_{\phi_f}(\tau)$ for

$\tau \in (\mathscr{H}_r)^d$ to $Z_{\phi_f}(g')$ for $g' \in \mathrm{U}(r,r)(\mathbb{A}_F)$ defined by

$$Z^G_{\phi_f}(g') := \sum_{x \in G(\mathbb{Q})\backslash \widehat{V}^r_E} \sum_{g \in G_x(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K^G_f} \omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi^d_+)(g^{-1}x) Z^G(x,g)_{K^G_f}$$
$$= \sum_{\substack{x \in K^G_f \backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2 \in Ex} \sum_{\substack{y_1 \in K^G_{f,x}\backslash x^\perp \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi^d_+)(x, y_1 + y_2) Z^G(y_1)_{K^G_{f,x}}.$$

Remark 4.2. The modularity of the generating series $Z_{\phi_f}(\tau)$ is equivalent to the left $\mathrm{U}(r,r)(F)$-invariance of the function $Z_{\phi_f}(g')$ on $\mathrm{U}(r,r)(\mathbb{A})$.

Hence, in the following, we prove the left $\mathrm{U}(r,r)$-invariance of $Z_{\phi_f}(g')$. First, we show the P_r-invariance of $Z^G_{\phi_f}(g')$ for any r. Second, for $r > 1$, we show the $w_{r,r-1}$-invariance of $Z_{\phi_f}(g')$, assuming w_1-invariance for $r = 1$. Finally, we show that $Z^G_{\phi_f}(g')$ is w_1-invariant for $r = 1$.

4.2. Invariance under the parabolic subgroup P_r. The elements $m(a)$ and $n(u)$ generate the parabolic subgroup $P_r(F) \subset \mathrm{U}(r,r)(F)$.

In the same way as in [9, Theorem 3.5(1)] or [10, Section 4.1], we can show the following invariance under $n(u)_f$ and $m(a)_f$:

$$\omega_f(n(u)_f g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^G(x)_{K^G_f} = \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^G(x)_{K^G_f},$$
$$\omega_f(m(a)_f g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^G(x)_{K^G_f} = \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(xa) Z^G(x)_{K^G_f},$$

for any $u \in \mathrm{Her}_r(F)$ and $a \in \mathrm{GL}_r(F)$. The first equation shows the $n(u)$-invariance of $Z_{\phi_f}(g')$. We shall prove that $Z_{\phi_f}(g')$ is $m(a)$-invariant as follows. We have $U(x) = U(xa)$, so $Z^G_{\phi_f}(x) = Z^G_{\phi_f}(xa)$. Therefore, combining the above calculation and the fact that $Z^G_{\phi_f}(x) = Z^G_{\phi_f}(xa)$, we conclude that

$$Z^G_{\phi_f}(\omega_f(m(a))g') = \sum_{\substack{x \in K^G_f \backslash \widehat{V}^r_E \\ \text{admissible}}} \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(xa) Z^G(xa)_{K^G_f}$$
$$= \sum_{\substack{x \in K^G_f \backslash \widehat{V}^r_E \\ \text{admissible}}} \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^G(x)_{K^G_f} = Z^G_{\phi_f}(g').$$

This shows that $Z_{\phi_f}(g')$ is invariant under the action of P_r.

4.3. Invariance under $w_{r,r-1}$ for $r > 1$. For the following discussion, we use the induction method of [9, proof of Theorem 3.5] and [12, Section 4.2]. Recall that

$$Z^G_{\phi_f}(g') = \sum_{\substack{x \in K^G_f \backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2 \in Ex} \sum_{\substack{y_1 \in K^G_{f,x}\backslash x^\perp \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi^d_+)(x, y_1 + y_2) Z^G(y_1)_{K^G_{f,x}}.$$

Hence,

$$Z^G_{\phi_f}(w_{r,r-1}g') = \sum_{\substack{x\in K^G_f\backslash \widehat{V}_E^{r-1}\\ \text{admissible}}} \sum_{y_2\in Ex} \sum_{\substack{y_1\in K^G_{f,x}\backslash x^\perp\\ \text{admissible}}} \omega_{\mathbb{A}}(w_{r,r-1})(\omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d))(x,y_1+y_2)Z^G(y_1)_{K^G_{f,x}}.$$

Now, from the definition of the Weil representation, we have

$$\omega_{\mathbb{A}}(w_{r,r-1})(\phi_f\otimes\varphi_+^d)(x,y) = (\phi_f\otimes\varphi_+^d)^y(x,y),$$

where $\phi^y(x,y)$ is the partial Fourier transformation with respect to the second coordinate. Applying this, we get

$$Z^G_{\phi_f}(w_{r,r-1}g') = \sum_{\substack{x\in K^G_f\backslash \widehat{V}_E^{r-1}\\ \text{admissible}}} \sum_{y_2\in Ex} \sum_{\substack{y_1\in K^G_{f,x}\backslash x^\perp\\ \text{admissible}}} (\omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d))^{y_1,y_2}(x,y_1+y_2)Z^G(y_1)_{K^G_{f,x}}.$$

For fixed x, applying the $r=1$ case (modularity of the generating series constructed by special divisors) to the special divisors $Z^G(y_1)_{K^G_{f,x}}$, we have

$$\sum_{\substack{y_1\in K^G_{f,x}\backslash x^\perp\\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d)^{y_1,y_2}(x,y_1+y_2)Z^G(y_1)_{K^G_{f,x}} = \sum_{\substack{y_1\in K^G_{f,x}\backslash x^\perp\\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d)^{y_2}(x,y_1+y_2)Z^G(y_1)_{K^G_{f,x}},$$

as a function of y_2. Note that $w_{1,0}=w_1$, and here we can use the w_1-modularity for $r=1$. Thus,

$$Z^G_{\phi_f}(w_{r,r-1}g') = \sum_{\substack{x\in K^G_f\backslash \widehat{V}_E^{r-1}\\ \text{admissible}}} \sum_{y_2\in Ex} \sum_{\substack{y_1\in K^G_{f,x}\backslash x^\perp\\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d)^{y_2}(x,y_1+y_2)Z^G(y_1)_{K^G_{f,x}}.$$

Here, for fixed x and y_2, by the Poisson summation formula for the function $\omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d)(x,y_1+y_2)$ on $y_2\in Ex\subset \mathbb{A}x$, we have

$$\sum_{y_2\in Ex} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d)^{y_2}(x,y_1+y_2)Z^G(y_1)_{K^G_{f,x}} = \sum_{y_2\in Ex} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d)(x,y_1+y_2)Z^G(y_1)_{K^G_{f,x}}.$$

This leads to

$$Z^G_{\phi_f}(w_{r,r-1}g') = \sum_{\substack{x\in K^G_f\backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2\in Ex} \sum_{\substack{y_1\in K^G_{f,x}\backslash x^\perp \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+)(x,y_1+y_2)Z^G(y_1)_{K^G_{f,x}},$$

which coincides with the definition of $Z^G_{\phi_f}(g')$. Therefore, we get

$$Z^G_{\phi_f}(w_{r,r-1}g') = Z^G_{\phi_f}(g').$$

This shows that $Z_{\phi_f}(g')$ is invariant under the action of $w_{r,r-1}$.

4.4. Invariance under w_1 for $r = 1$. We use Liu's proof of [9, Theorem 3.5]. Now, $\mathrm{U}(1)\times\mathrm{U}(1)$ is the maximal compact subgroup of $\mathrm{U}(1,1)$, and $\mathrm{SL}_2(\mathbb{A}_{F,f})(\mathrm{U}(1)\times\mathrm{U}(1))(\mathbb{A}_{F,f}) = \mathrm{U}(1,1)(\mathbb{A}_{F,f})$. Therefore, we reduce the problem to proving that $Z^G_{\phi_f}(w_1g') = Z^G_{\phi_f}(g')$ for all $g'\in\mathrm{SL}_2(\mathbb{A}_F)$. By [9, Corollary 3.4] and [9, proof of Lemma 3.6], it suffices to prove $Z^H_{\phi_f}(w_1g') = Z^H_{\phi_f}(g')$. However, this follows from [7] or [10] under the Beilinson–Bloch conjecture for orthogonal Shimura varieties. This finishes the proof of Theorem 4.1.

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Abstract (will appear on the journal's web site only)

We study the modularity of the generating series of special cycles on unitary Shimura varieties over CM-fields of degree $2d$ associated with a Hermitian form in $n+1$ variables whose signature is $(n,1)$ at e real places and $(n+1,0)$ at the remaining $d-e$ real places for $1 \leq e < d$. For $e=1$, Liu proved the modularity, and Xia showed the absolute convergence of the generating series. On the other hand, Bruinier constructed regularized theta lifts on orthogonal groups over totally real fields and proved the modularity of special divisors on orthogonal Shimura varieties. By using Bruinier's result, we work on the problem for $e=1$ and give another proof of Liu's theorem [Algebra Number Theory 5 (2011)]. For $e>1$, we prove that the generating series of special cycles of codimension er in the Chow group is a Hilbert-Hermitian modular form of weight $n+1$ and genus r, assuming the Beilinson–Bloch conjecture for orthogonal Shimura varieties. Our result is a generalization of Kudla's modularity conjecture, solved by Liu unconditionally when $e=1$.